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Controllability results for cascade systems of m coupled parabolic PDEs by one control force

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Abstract. In this paper we will analyze the controllability properties of a linear coupled parabolic system of m equations when a unique distributed control is exerted on the system. We will see that, when a cascade system is considered, we can prove a global Carleman inequality for the adjoint system which bounds the global integrals of the variable $\varphi = (\varphi_1, \dots, \varphi_m)^*$ in terms of a unique localized variable. As a consequence, we will obtain the null controllability property for the system with one control force.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set with boundary $\partial\Omega$ of class C^2. Let $\omega \subset \Omega$ be a nonempty open subset and assume $T > 0$. All along this work we will denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. For $m \geq 1$ given, we consider the parabolic linear system

$$\begin{cases} \partial_t y_1 - L_1 y_1 + \displaystyle\sum_{j=1}^m B_{1j} \cdot \nabla y_j + \sum_{j=1}^m a_{1j} y_j = v 1_\omega \ \text{ in } Q = \Omega \times (0,T), \\ \partial_t y_2 - L_2 y_2 + \displaystyle\sum_{j=1}^m B_{2j} \cdot \nabla y_j + \sum_{j=1}^m a_{2j} y_j = 0 \ \text{ in } Q, \\ \dots \\ \partial_t y_m - L_m y_m + \displaystyle\sum_{j=1}^m B_{mj} \cdot \nabla y_j + \sum_{j=1}^m a_{mj} y_j = 0 \ \text{ in } Q, \\ y_i = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \quad y_i(x,0) = y_{0,i}(x) \text{ in } \Omega, \quad 1 \leq i \leq m, \end{cases}$$

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where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q)$, $B_{ij} = B_{ij}(x,t) \in L^\infty(Q)^N$ $(1 \le i,j \le m)$, $y_{0,i} \in L^2(\Omega)$ $(1 \le i \le m)$ and L_k is, for every $1 \le k \le m$, the self-adjoint second order operator

$$L_k y(x,t) = \sum_{i,j=1}^{N} \partial_i \left(\alpha_{ij}^k(x,t) \partial_j y(x,t) \right), \tag{1}$$

$(\partial_i = \partial/\partial_{x_i})$ where

$$\alpha_{ij}^k \in W^{1,\infty}(Q), \quad \alpha_{ij}^k(x,t) = \alpha_{ji}^k(x,t) \text{ a.e. in } Q, \quad 1 \le i,j \le N, \quad 1 \le k \le m, \tag{2}$$

and, for all $k : 1 \le k \le m$, the coefficients α_{ij}^k satisfy

$$\max_{i,j,k} ||\alpha_{ij}^k||_{W^{1,\infty}} = \widetilde{M}_0, \quad \sum_{i,j=1}^{N} \alpha_{ij}^k(x,t)\xi_i\xi_j \ge \widetilde{a}_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. in } Q, \tag{3}$$

for positive constants $\widetilde{M}_0$ and $\widetilde{a}_0$.

Equivalently, the previous system can be written as

$$\begin{cases} \partial_t y - Ly + B \cdot \nabla y + Ay = Dv1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(x,0) = y_0(x) \text{ in } \Omega, \end{cases} \tag{4}$$

where L is the matrix operator given by $L = \operatorname{diag}(L_1, ..., L_m)$, $y = (y_i)_{1 \le i \le m}$ is the state, 1_ω is the characteristic function of the nonempty set ω and $\nabla y = (\nabla y_i)_{1 \le i \le m}$, and where

$$\begin{cases} y_0 = (y_{0,i})_{1 \le i \le m} \in L^2(\Omega)^m, \quad A(x,t) = (a_{ij}(x,t))_{1 \le i,j \le m} \in L^\infty(Q)^{m^2}, \\ B(x,t) = (B_{ij}(x,t))_{1 \le i,j \le m} \in L^\infty(Q)^{Nm^2} \text{ and } D \equiv e_1 = (1,0,...,0)^* \end{cases}$$

are given. Let us observe that, for each $y_0 \in L^2(\Omega)^m$ and $v \in L^2(Q)$, system (4) admits a unique weak solution $y \in L^2(0,T;H_0^1(\Omega)^m) \cap C^0([0,T];L^2(\Omega)^m)$.

The main goal of this paper is to analyze the controllability properties of (4). It will be said that (4) is null controllable at time T if for every $y_0 \in L^2(\Omega)^m$ there exists a control $v \in L^2(Q)$ such that the solution y of (4) satisfies

$$y_i(\cdot,T) = 0 \quad \text{in } \Omega, \quad \forall i : 1 \le i \le m. \tag{5}$$

When $m = 1$ (one equation and one control force) the null controllability of parabolic problems has been studied by several authors (see for instance [18], [17], [4], [10], ...). We also point out [14] and [15] where the null controllability of system (4) at time T was established for every $T > 0$, $A \in L^\infty(Q)$, $B \in L^\infty(Q)^N$ and $\omega \subset \Omega$, using a global Carleman inequality for the solutions of the corresponding adjoint problem.

There are few results on null controllability of system (4) when $m > 1$ and most of them are proved for $m = 2$. In [19] and [5] the authors consider a nonlinear system of two heat equations, one of them being forward and the other one backward in time, and show the null controllability of this system with sublinear nonlinearities ([19]) or slightly superlinear nonlinearities ([5]). In [1] and [2], the authors give a null controllability result for a phase-field system and for reaction-diffusion systems (two nonlinear heat equations). The results in [1] and [2] have been generalized in [12] (see also [11]) in two directions: on the one hand, there are not restrictions on the dimension N, and on the other hand, the authors consider nonlinearities which depend on the gradient of the state. Finally, in [8] the authors prove a result of local exact controllability to the trajectories for the Boussinesq system ($N+1$ equations) when N (or $N-1$) distributed controls are exerted on the system.

All previous works have a common point: they deal with cascade systems. In this work we want to generalize these works to the case of a general cascade system of m linear parabolic equations. To this end, we will suppose that A and B have the structure

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ 0 & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{m,m-1} & a_{mm} \end{pmatrix}, \; B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ 0 & B_{22} & \dots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{mm} \end{pmatrix} \tag{6}$$

with $a_{ij} \in L^\infty(Q)$, $B_{ij} \in L^\infty(Q)^N$ $(1 \le i \le j \le m)$ and $a_{i,i-1} \in L^\infty(Q)$ $(2 \le i \le m)$, for an open set $\omega_0 \subset \omega$, satisfy

$$a_{i,i-1} \ge a_0 > 0 \text{ or } -a_{i,i-1} \ge a_0 > 0 \text{ in } \omega_0 \times (0,T), \ \forall i : 2 \le i \le m. \tag{7}$$

In order to study the null controllability of system (4), we will consider the corresponding adjoint problem which, under assumption (6) (cascade system), has the form

$$\begin{cases} -\partial_t \varphi_i - L_1 \varphi_i - \displaystyle\sum_{j=1}^{i} [\nabla \cdot (B_{ji} \varphi_j) - a_{ji} \varphi_j] = -a_{i+1,i} \varphi_{i+1} \ \text{ in } Q, \\ \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \le i \le m-1), \\ -\partial_t \varphi_m - L_m \varphi_m - \displaystyle\sum_{j=1}^{m} [\nabla \cdot (B_{jm} \varphi_j) - a_{jm} \varphi_j] = 0 \ \text{ in } Q, \\ \varphi_i = 0 \text{ on } \Sigma, \quad \varphi_i(x,T) = \varphi_{0,i}(x) \text{ in } \Omega, \quad 1 \le i \le m, \end{cases} \tag{8}$$

where $\varphi_{0,i} \in L^2(\Omega)$ $(1 \le i \le m)$.

It is well known that the null controllability of system (4) (with L^2-controls) is equivalent to the existence of a constant $C_0 > 0$ such that the so-called *observability inequality*

$$||\varphi(\cdot,0)||^2_{L^2(\Omega)^m} \le C_0 \iint_{\omega \times (0,T)} |\varphi_1(x,t)|^2 \, dx \, dt. \tag{9}$$

holds for every solution $\varphi = (\varphi_1, \dots, \varphi_m)^*$ to (8). Let us observe that in (9) we are estimating the L^2-norm of $\varphi(\cdot, 0)$ by means of the L^2-norm of the first component of φ localized in $\omega \times (0,T)$. We will prove inequality (9) as a consequence of a global Carleman inequality for the adjoint system (8). This inequality is established in our first main result:

Theorem 1.1. *Let us suppose that* L_k, $A \in L^\infty(Q)^{m^2}$ *and* $B \in L^\infty(Q)^{Nm^2}$ *are given by* (1) *and* (6) *and satisfy* (2), (3) *and* (7). *Let* $M_0 = \max_{2 \le i \le m} ||a_{i,i-1}||_\infty$. *Then, there exist a positive function* $\alpha_0 \in C^2(\overline{\Omega})$ *(only depending on* Ω *and* ω_0*), two positive constants* C_0 *(only depending on* Ω, ω_0, m, $\widetilde{a}_0$, $\widetilde{M}_0$, a_0 *and* M_0*) and* $\sigma_0 = \sigma_0(\Omega, \omega_0, m, \widetilde{a}_0, \widetilde{M}_0, M_0)$ *and* $l \ge 3$ *(only depending on* m*) such that, for every* $\varphi_0 \in L^2(Q)^m$, *the solution* φ *to* (8) *satisfies*

$$\sum_{i=1}^m \mathcal{I}(3(m+1-i), \varphi_i) \le C_0 s^l \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^l |\varphi_1|^2, \tag{10}$$

$\forall s \ge s_0 = \sigma_0 \left[T + T^2 + T^2 \max_{i \le j} \left(||a_{ij}||_\infty^{\frac{2}{3(j-i)+3}} + ||B_{ij}||_\infty^{\frac{2}{3(j-i)+1}} \right) \right]$. *In inequality* (10), $\alpha(x,t)$, $\gamma(t)$ *and* $\mathcal{I}(d,z)$ *are given by:* $\alpha(x,t) \equiv \alpha_0(x)/t(T-t)$, $\gamma(t) \equiv (t(T-t))^{-1}$ *and*

$$\mathcal{I}(d,z) \equiv s^{d-2} \iint_Q e^{-2s\alpha} \gamma(t)^{d-2} |\nabla z|^2 + s^d \iint_Q e^{-2s\alpha} \gamma(t)^d |z|^2.$$

We will prove Theorem 1.1 from the corresponding global Carleman inequality satisfied by the solutions to the heat equation with a right hand side in the space $L^2(0,T; H^{-1}(\Omega))$ (see [15]). To prove Theorem 1.1 we will follow the same argument given in [19] and [12] and which allows to show (10) when $m = 2$.

As a consequence of Theorem 1.1 we will obtain the null controllability at time T of system (5). This is our second main result:

Theorem 1.2. *Under assumptions of Theorem* 1.1, *given* $y_0 \in L^2(\Omega)^m$, *there exists a control* $v \in L^2(Q)$ *such that* $\operatorname{Supp} v \subset \overline{\omega}_0 \times [0,T]$ *and the corresponding solution* y *to* (4) *satisfies* (5). *Moreover, the control* v *can be chosen such that*

$$||v||^2_{L^2(Q)} \le e^{C\,\mathcal{H}} \sum_{i=1}^m ||y_{0,i}||^2_{L^2(\Omega)}, \tag{11}$$

with C *a positive constant, only depending on* Ω, ω_0, $\widetilde{a}_0$, $\widetilde{M}_0$, a_0 *and* M_0, *and*

$$\mathcal{H} \equiv 1 + T + \frac{1}{T} + \max_{i \le j} \left(||a_{ij}||_\infty^{\frac{2}{3(j-i)+3}} + ||B_{ij}||_\infty^{\frac{2}{3(j-i)+1}} + T \left(||a_{ij}||_\infty + ||B_{ij}||^2_\infty \right) \right).$$

Let us remark that, thanks to the cascade structure of the coupling matrices A and B (see (6) and (7)), we can control the system (4) (m functions) by means of one control force $v \in L^2(Q)$ (exerted in the right hand side of the first equation of the system). For general complete matrices, this is not possible, in general. Indeed, let us consider $m = 3$, $L_1 = L_2 = L_3 = \Delta$ (which, evidently satisfy (2) and (3)), $B = 0$ and $A \in \mathcal{L}(\mathbb{R}^3)$ a constant matrix given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions and let ϕ_1 be the associated eigenfunction with $||\phi_1||_{L^2(\Omega)} = 1$. If we now consider $\varphi_0(x,t) = (0,-1,1)^*\phi_1(x) \in L^2(\Omega)^m$ in (8), it is not difficult to see that the corresponding solution to (8) is given by $\varphi(x,t) = (0,-1,1)^* e^{(\lambda_1+1)(t-T)}\phi_1(x)$ which, evidently, does not satisfy inequality (10). In fact, the observability inequality (9) is also false and therefore, system (4) is not null controllable. The null controllability problem of system (4) for general coupling matrices A and B is nowadays widely open.

Another important point to be underlined is that in our controllability problem the control is exerted on a little part ω of the set Ω (distributed control). The controllability properties of cascade systems like (4) can fail if boundary controls are considered instead of distributed controls. In [7] the boundary controllability of a cascade system of two parabolic equations is studied and it was found that even the boundary approximate controllability of the system is not in general true. To be precise, let us consider the controlled system ($N = 1$, $m = 2$)

$$\begin{cases} \partial_t y - Ly + Ay = 0 \ \text{ in } Q = (0,1) \times (0,T), \\ y = Dv \ \text{ on } \{0\} \times (0,T), \quad y = 0 \ \text{ on } \{1\} \times (0,T) \\ y(\cdot,0) = y_0 \ \text{ in } (0,1), \end{cases}$$

where $v \in L^2(0,T)$ is the control,

$$Ly = \begin{pmatrix} y_{xx} & 0 \\ 0 & \nu y_{xx} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\nu > 0$ (which evidently satisfy (2), (3) and (7)). Then, in [7] and by means of a simple counterexample, it is proved that this system is not approximately controllable if $\sqrt{\nu} \in \mathbb{Q} \setminus \{1\}$.

The rest of the work is organized as follows. In Section 2 we will prove the Carleman inequality stated in Theorem 1.1. Theorem 1.2 will be proved in Section 3. Finally, we will devote Section 4 to give some remarks and additional results.

2. The global Carleman inequality. Proof of Theorem 1.1

We will devote this section to prove Theorem 1.1. To this end, we will suppose that the coupling matrices $A \in L^\infty(Q)^{m^2}$ and $B \in L^\infty(Q)^{Nm^2}$ are given by (6) and satisfy (7). The basic tool we will use is a global Carleman inequality satisfied by the solutions to

$$\begin{cases} -\partial_t z - L_0 z = F \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \quad z(x,T) = z^0(x) \quad \text{in } \Omega, \end{cases} \tag{12}$$

with $z^0 \in L^2(\Omega)$ and $F = F_0 + \sum_{i=1}^N \frac{\partial F_i}{\partial x_i}$ with $F_i \in L^2(Q)$, $i = 0, 1, \dots, N$, and L_0 is given by

$$L_0 y(x,t) = \sum_{i,j=1}^N \partial_i \left(\alpha_{ij}^0(x,t) \partial_j y(x,t) \right)$$

where the coefficients $\alpha_{ij}^0 \in W^{1,\infty}(Q)$ $(1 \le i, j \le N)$ satisfy $\alpha_{ij}^0(x,t) = \alpha_{ji}^0(x,t)$ a.e. in Q and

$$\max_{i,j} ||\alpha_{ij}^0||_{W^{1,\infty}} = \widehat{M}_0, \quad \sum_{i,j=1}^N \alpha_{ij}^0(x,t)\xi_i\xi_j \ge \widehat{a}_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. in } Q,$$

for positive constants $\widehat{a}_0$ and $\widehat{M}_0$. One has:

Lemma 2.1. *Let $\mathcal{B} \subset \Omega$ be a nonempty open subset and $d \in \mathbb{R}$. Then, there exist a function $\beta_0 \in C^2(\overline{\Omega})$ (only depending on Ω and $\mathcal{B}$) and two positive constants $\widetilde{C}_0$ and $\widetilde{\sigma}_0$ (which only depend on Ω, $\mathcal{B}$, $\widehat{a}_0$, $\widehat{M}_0$ and d) such that, for every $z^0 \in L^2(\Omega)$, the solution z to* (12) *satisfies*

$$\begin{cases} s^{d-2} \displaystyle\iint_Q e^{-2s\beta} \gamma(t)^{d-2} |\nabla z|^2 + s^d \iint_Q e^{-2s\beta} \gamma(t)^d |z|^2 \le \widetilde{C}_0 \Bigg(\mathcal{L}_{\mathcal{B}}(d,z) \\ \quad + s^{d-3} \displaystyle\iint_Q e^{-2s\beta} \gamma(t)^{d-3} |F_0|^2 + s^{d-1} \sum_{i=1}^N \iint_Q e^{-2s\beta} \gamma(t)^{d-1} |F_i|^2 \Bigg), \end{cases} \tag{13}$$

for all $s \ge \widetilde{s}_0 = \widetilde{\sigma}_0 \, (T + T^2)$. In (13)*, $\mathcal{L}_{\mathcal{B}}(d,z)$ and the functions β and γ are given by*

$$\mathcal{L}_{\mathcal{B}}(d,z) \equiv s^d \iint_{\mathcal{B} \times (0,T)} e^{-2s\alpha} \gamma(t)^d |z|^2, \quad \beta(x,t) = \frac{\beta_0(x)}{t(T-t)}, \ (x,t) \in Q,$$

and $\gamma(t) = (t(T-t))^{-1}$, $t \in (0,T)$. □

The proof of this result can be found in [15] although the authors do not specify the way the constant $\widetilde{s}_0$ depends on T. This explicit dependence can be obtained arguing as in [9] (also see [6]).

Proof of Theorem 1.1*:* Given $\omega_0 \subset \omega$, we choose $\omega_1 \subset\subset \omega_0$. Let $\alpha_0 \in C^2(\overline{\Omega})$ be the function provided by Lemma 2.1 and associated to Ω and $\mathcal{B} \equiv \omega_1$, and let $\alpha(x,t)$ the function given by $\alpha(x,t) = \alpha_0(x)/t(T-t)$. We will do the proof in two steps:

Step 1. Let $\varphi = (\varphi_1, \dots, \varphi_m)^*$ be the solution to (8) associated to $\varphi_0 \in L^2(\Omega)^m$. We begin applying inequality (13) with $\mathcal{B} = \omega_1$ to each function φ_i ($1 \le i \le m$) with $L_0 \equiv L_i$, $d = 3(m+1-i)$ and

$$F \equiv \sum_{j=1}^{i} [\nabla \cdot (B_{ji}\varphi_j) - a_{ji}\varphi_j] - a_{i+1,i}\varphi_{i+1},$$

if $1 \le i \le m-1$, and $F \equiv \sum_{j=1}^{m} [\nabla \cdot (B_{jm}\varphi_j) - a_{jm}\varphi_j]$, for $i = m$, obtaining

$$\left\{\begin{aligned} \mathcal{I}(3(m+1-i), \varphi_i) \le \widehat{C}_0 \Bigg(& \mathcal{L}_{\omega_1}(3(m+1-i), \varphi_i) + M_0\, \mathcal{I}(3(m-i), \varphi_{i+1}) \\ & + \sum_{j=1}^{i} s^{3(m-i)} ||a_{ji}||_\infty^2 \iint_Q e^{-2s\alpha} \gamma(t)^{3(m-i)} |\varphi_j|^2 \\ & + \sum_{j=1}^{i} s^{3(m-i)+2} ||B_{ji}||_\infty^2 \iint_Q e^{-2s\alpha} \gamma(t)^{3(m-i)+2} |\varphi_j|^2 \Bigg), \end{aligned}\right.$$

for every $s \ge \widehat{s}_0 = \widehat{\sigma}_0\,(T+T^2)$, with $\widehat{C}_0$ and $\widehat{\sigma}_0$ two positive constant only depending on Ω, ω_1, $\widetilde{a}_0$, $\widetilde{M}_0$ (and m) (in the previous inequality we have taken $\varphi_{i+1} \equiv 0$ when $i = m$). Now, it is not difficult to see that, for a new constant C (depending on Ω, ω_1, $\widetilde{a}_0$, $\widetilde{M}_0$ and M_0), one has:

$$\left\{\begin{aligned} \sum_{i=1}^{m} \mathcal{I}(3(m+1-i), \varphi_i) \le C \Bigg(& \sum_{i=1}^{m} \mathcal{L}_{\omega_1}(3(m+1-i), \varphi_i) \\ & + \sum_{i=1}^{m} \sum_{j=1}^{i} s^{3(m-i)} ||a_{ji}||_\infty^2 \iint_Q e^{-2s\alpha} \gamma(t)^{3(m-i)} |\varphi_j|^2 \\ & + \sum_{i=1}^{m} \sum_{j=1}^{i} s^{3(m-i)+2} ||B_{ji}||_\infty^2 \iint_Q e^{-2s\alpha} \gamma(t)^{3(m-i)+2} |\varphi_j|^2 \Bigg), \quad \forall s \ge \widehat{s}_0. \end{aligned}\right.$$

Finally, we can get rid of the two last sums in the previous inequality if we take into account that $t(T-t) \le T^2/4$ in $(0,T)$ and we take

$$s \ge s_0 = \sigma_0 \left(T + T^2 + T^2 \max_{i \le j} \left(||a_{ij}||_\infty^{\frac{2}{3(j-i)+3}} + ||B_{ij}||_\infty^{\frac{2}{3(j-i)+1}} \right) \right), \tag{14}$$

with $\sigma_0 = \sigma_0(\Omega, \omega_0, \widetilde{a}_0, \widetilde{M}_0, M_0) > 0$, obtaining, for a positive constants $C_1 = C_1(\Omega, \omega_0, \widetilde{a}_0, \widetilde{M}_0, M_0)$,

$$\sum_{i=1}^{m} \mathcal{I}(3(m+1-i), \varphi_i) \le C_1 \left(\sum_{i=1}^{m} \mathcal{L}_{\omega_1}(3(m+1-i), \varphi_i) \right), \quad \forall s \ge s_0. \tag{15}$$

Step 2. We will see that, thanks to assumptions (6) and (7), we can eliminate in (15) the local terms $\mathcal{L}_{\omega_1}(3(m+1-i), \varphi_i)$ for $2 \le i \le m$. In order to carry this process out, we will need the following result:

Lemma 2.2. *Under assumptions of Theorem* 1.1 *and given $l \in \mathbb{N}$, $\varepsilon > 0$, $k \in \{2, ..., m\}$ and two open sets $\mathcal{O}_0$ and $\mathcal{O}_1$ such that $\omega_1 \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \omega_0$, there exist a positive constant C_k (only depending on Ω, $\mathcal{O}_0$, $\mathcal{O}_1$, $\widetilde{a}_0$, $\widetilde{M}_0$, a_0 and M_0) and $l_{kj} \in \mathbb{N}$, $1 \leq j \leq k-1$ (only depending on l, m, k and j), such that, if φ is the solution to* (8) *associated to $\varphi_0 \in L^2(Q)^m$ and $s \geq s_0$, one has*

$$\left\{ \begin{array}{l} \mathcal{L}_{\mathcal{O}_1}(l, \varphi_k) \leq \varepsilon \left[\mathcal{I}(3(m+1-k), \varphi_k) + \mathcal{I}(3(m-k), \varphi_{k+1}) \right] \\ \qquad\qquad + C_k \left(1 + \dfrac{1}{\varepsilon} \right) \displaystyle\sum_{j=1}^{k-1} \mathcal{L}_{\mathcal{O}_0}(l_{kj}, \varphi_j). \end{array} \right. \tag{16}$$

(In this inequality we have taken $\varphi_{k+1} \equiv 0$ when $k = m$). □

We will finish the proof of Theorem 1.1 showing that it is an easy consequence of Lemma 2.2. To this end, we consider open sets $\widetilde{\mathcal{O}}_i \subset \omega_0$, with $2 \leq i \leq m$, such that $\omega_1 \subset\subset \widetilde{\mathcal{O}}_m \subset\subset \widetilde{\mathcal{O}}_{m-1} \subset\subset \cdots \subset\subset \widetilde{\mathcal{O}}_2 \subset \omega_0$ and we begin by applying formula (16) for $\mathcal{O}_1 = \omega_1$, $\mathcal{O}_0 = \widetilde{\mathcal{O}}_m$, $k = m$, $l = 3$ and $\varepsilon = 1/2C_1$ (with C_1 the constant appearing in (15)). Thus, from (15), we obtain

$$\sum_{i=1}^{m} \mathcal{I}(3(m+1-i), \varphi_i) \leq \widetilde{C}_m \left(\sum_{i=1}^{m-1} \mathcal{L}_{\widetilde{\mathcal{O}}_m}(l_{mi}, \varphi_i) \right), \tag{17}$$

for all $s \geq s_0$, with $\widetilde{C}_m$ a new positive constant only depending on Ω, ω_1, $\widetilde{\mathcal{O}}_m$, $\widetilde{a}_0$, $\widetilde{M}_0$, a_0 and M_0. Observe that in (17) we have eliminated in the right hand side the term that depends on φ_m. We can go on applying (16) for $\mathcal{O}_1 = \widetilde{\mathcal{O}}_m$, $\mathcal{O}_0 = \widetilde{\mathcal{O}}_{m-1}$, $k = m-1$, $l = l_{m,m-1}$ and $\varepsilon = 1/2\widetilde{C}_m$ and eliminate in (17) the local term $\mathcal{L}_{\widetilde{\mathcal{O}}_m}(l_{m,m-1}, \varphi_{m-1})$. By (a finite) iteration of this argument we obtain (10). □

Proof of Lemma 2.2*:* For the proof of this result, we will reason out as in [19] and [12]. All along the proof, C will be a generic constant that may depend on Ω, $\mathcal{O}_0$, $\mathcal{O}_1$, $\widetilde{a}_0$, $\widetilde{M}_0$, a_0 and M_0, and also we will assume that s satisfies $s \geq s_0$, with s_0 given by (14). In particular, $s \geq C(T + T^2)$ and then, for $\mu, \nu \in \mathbb{Z}$, $\nu \leq \mu$, and for every $(x,t) \in Q$, one has

$$\left\{ \begin{array}{l} [s\gamma(t)]^\nu \leq C\, [s\gamma(t)]^\mu, \quad \left| \nabla [s^\nu e^{-2s\alpha} \gamma(t)^\nu] \right| \leq C s^{\nu+1} e^{-2s\alpha} \gamma(t)^{\nu+1}, \\ \left| \partial_t [s^\nu e^{-2s\alpha} \gamma(t)^\nu] \right| + \sum_{i,j\geq 1}^{N} \left| \partial^2_{ij} [s^\nu e^{-2s\alpha} \gamma(t)^\nu] \right| \leq C s^{\nu+2} e^{-2s\alpha} \gamma(t)^{\nu+2}. \end{array} \right. \tag{18}$$

Given $\omega_1 \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \omega_0$, we consider $\xi_0 \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \xi_0 \leq 1$ in $\mathbb{R}^N$, $\xi_0 \equiv 1$ in $\mathcal{O}_1$, Supp $\xi_0 \subset \mathcal{O}_0$ and

$$\frac{\Delta \xi_0}{\xi_0^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\nabla \xi_0}{\xi_0^{1/2}} \in L^\infty(\Omega)^N. \tag{19}$$

Let us consider $k \in \{2, ..., m\}$. The coefficient $a_{k,k-1}$ satisfies assumption (7) and, for simplicity, let us assume $a_{k,k-1} \geq a_0$ in $\omega_0 \times (0,T)$. We fix $l \in \mathbb{N}$ and take $u = s^l e^{-2s\alpha}\gamma(t)^l$. We multiply the equation satisfied by φ_{k-1} by $u\xi_0\varphi_k$ and integrate in Q. We get,

$$\left\{\begin{array}{l} a_0\,\mathcal{L}_{\mathcal{O}_1}(l,\varphi_k) \leq \tilde{L}(l,\varphi_k) = \displaystyle\int_0^T \langle \partial_t\varphi_{k-1} + L_{k-1}\varphi_{k-1}, u\xi_0\varphi_k\rangle\,dt \\ -\displaystyle\sum_{j=1}^{k-1}\iint_Q u\xi_0\varphi_k a_{j,k-1}\varphi_j + \sum_{j=1}^{k-1}\int_0^T \langle\nabla\cdot(B_{j,k-1}\varphi_j), u\xi_0\varphi_k\rangle\,dt = \sum_{n=1}^4 K_n, \end{array}\right. \tag{20}$$

where $\tilde{L}(l,\varphi_k) = s^l \displaystyle\iint_Q e^{-2s\alpha}\gamma(t)^l\,\xi_0\,a_{k,k-1}\,|\varphi_k|^2$ and by means of $\langle\cdot,\cdot\rangle$ we are denoting the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Integrating by parts with respect to t and having in mind the equation satisfied by φ_k (see (8)), we get

$$\begin{aligned} K_1 &= \int_0^T \langle\partial_t\varphi_{k-1}, u\xi_0\varphi_k\rangle = -\iint_Q u_t\xi_0\varphi_k\varphi_{k-1} - \int_0^T \langle\partial_t\varphi_k, u\xi_0\varphi_{k-1}\rangle \\ &= -\iint_Q u_t\xi_0\varphi_k\varphi_{k-1} + \int_0^T \langle L_k\varphi_k + \sum_{j=1}^k (\nabla\cdot(B_{jk}\varphi_j) - a_{jk}\varphi_j)\,, u\xi_0\varphi_{k-1}\rangle \\ &- \iint_Q u\xi_0\varphi_{k-1}a_{k+1,k}\varphi_{k+1} = \sum_{n=1}^5 K_1^{(n)}. \end{aligned}$$

Let us remark that, when $k = m$, in the previous equality the last term $K_1^{(5)}$ does not appear. Taking into account (18) and (7), if $s \geq C(T + T^2)$, we obtain

$$\begin{aligned} |K_1^{(1)}| = \left|\iint_Q u_t\xi_0\varphi_k\varphi_{k-1}\right| &\leq Cs^{l+2}\iint_Q e^{-2s\alpha}\gamma(t)^{l+2}\xi_0|\varphi_k||\varphi_{k-1}| \\ &\leq \delta\tilde{L}(l,\varphi_k) + \frac{C}{\delta}s^{l+4}\iint_Q e^{-2s\alpha}\gamma(t)^{l+4}a_{k,k-1}^{-1}\xi_0|\varphi_{k-1}|^2 \\ &\leq \delta\tilde{L}(l,\varphi_k) + \frac{C}{\delta}s^{l+4}\iint_Q e^{-2s\alpha}\gamma(t)^{l+4}\xi_0|\varphi_{k-1}|^2, \end{aligned}$$

with $\delta > 0$ to be chosen.

Observe that, integrating by parts in $K_1^{(2)}$, we get

$$\left\{\begin{array}{l} K_1^{(2)} = -\displaystyle\sum_{i,j\geq 1}^N \left(\iint_Q \varphi_{k-1}\xi_0\,\alpha_{ij}^k\,\partial_i u\,\partial_j\varphi_k + \iint_Q \varphi_{k-1}u\,\alpha_{ij}^k\,\partial_i\xi_0\,\partial_j\varphi_k\right. \\ \qquad\left. + \displaystyle\iint_Q u\xi_0\,\alpha_{ij}^k\,\partial_i\varphi_{k-1}\,\partial_j\varphi_k\right). \end{array}\right.$$

From (18) and (19), we also have:

$$
\left\{
\begin{aligned}
|K_1^{(2)}| \le{} & Cs^{l+1} \iint_Q e^{-2s\alpha}\gamma(t)^{l+1}\xi_0|\varphi_{k-1}||\nabla\varphi_k| \\
& + Cs^l \iint_Q e^{-2s\alpha}\gamma(t)^l \xi_0^{1/2}|\varphi_{k-1}||\nabla\varphi_k| + \iint_Q u\xi_0|\nabla\varphi_{k-1}||\nabla\varphi_k| \\
& \le \frac{\varepsilon}{12} s^{3(m-k)+1} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-k)+1}|\nabla\varphi_k|^2 \\
& + \frac{C}{\varepsilon}\left(s^{n+2} \iint_Q e^{-2s\alpha}\gamma(t)^{n+2}\xi_0|\varphi_{k-1}|^2 \right. \\
& \left. + s^n \iint_Q e^{-2s\alpha}\gamma(t)^n\xi_0|\nabla\varphi_{k-1}|^2 \right),
\end{aligned}
\right.
\tag{21}
$$

with $n = 2l - 1 - 3(m-k)$. For $K_1^{(3)}$ we get:

$$
\begin{aligned}
K_1^{(3)} &= \sum_{j=1}^{k-1} \int_0^T \langle \nabla\cdot(B_{jk}\varphi_j), u\xi_0\varphi_{k-1}\rangle\, dt + \int_0^T \langle \nabla\cdot(B_{kk}\varphi_k), u\xi_0\varphi_{k-1}\rangle\, dt \\
&= -\sum_{j=1}^{k-1} \iint_Q [\nabla(u\xi_0)\cdot B_{jk}\varphi_{k-1}\varphi_j + u\xi_0 B_{jk}\cdot\nabla\varphi_{k-1}\varphi_j] \\
&- \iint_Q \nabla(u\xi_0)\cdot B_{kk}\varphi_{k-1}\varphi_k - \iint_Q u\xi_0 B_{kk}\cdot\nabla\varphi_{k-1}\varphi_k \\
&= M_1 + M_2 + M_3.
\end{aligned}
$$

Using (7), (18), (19) and taking into account that, if $s \ge C||B_{kk}||_\infty^2 T^2$, then, $||B_{kk}||_\infty^2 \le Cs\gamma(t)$, it is not difficult to see that, for $\delta > 0$, one has

$$
\begin{aligned}
|M_2 + M_3| \le \delta\tilde{L}(l,\varphi_k) + \frac{C}{\delta}\Bigg(& s^{l+3} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{l+3}|\varphi_{k-1}|^2 \\
& + s^{l+1} \iint_Q e^{-2s\alpha}\gamma(t)^{l+1}\xi_0|\nabla\varphi_{k-1}|^2 \Bigg).
\end{aligned}
$$

Now, using again (19) and (18), we get $(n = 2l - 1 - 3(m-k))$

$$
\begin{aligned}
|M_1| \le C\Bigg(& s^n \iint_Q e^{-2s\alpha}\gamma(t)^n\xi_0|\nabla\varphi_{k-1}|^2 \\
& + s^{n+2} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{n+2}|\varphi_{k-1}|^2 \Bigg) \\
& + \frac{1}{2}\sum_{j=1}^{k-1} s^{3(m-k)+1} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-k)+1}\xi_0|B_{jk}|^2|\varphi_j|^2.
\end{aligned}
$$

Since $s \geq CT^2 ||B_{jk}||_\infty^{\frac{2}{3(k-j)+1}}$, we also have $||B_{jk}||_\infty^2 \leq C\left(s/T^2\right)^{3(k-j)+1} \leq C\left(s\gamma(t)\right)^{3(k-j)+1}$. This implies that,

$$
\begin{aligned}
|M_1| \leq C\Bigg(& s^{n+2} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{n+2}|\varphi_{k-1}|^2 \\
& + s^n \iint_Q e^{-2s\alpha}\gamma(t)^n \xi_0 |\nabla\varphi_{k-1}|^2 \Bigg) \\
& + C\sum_{j=1}^{k-1} s^{3(m-j)+2} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-j)+2}\xi_0|\varphi_j|^2.
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
|K_1^{(3)}| \leq \delta\tilde{L}(l,\varphi_k) + \frac{C}{\delta}\Bigg(& s^{l+3} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{l+3}|\varphi_{k-1}|^2 \\
& + s^{l+1} \iint_Q e^{-2s\alpha}\gamma(t)^{l+1}\xi_0|\nabla\varphi_{k-1}|^2 \Bigg) \\
& + C\Bigg(s^{n+2} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{n+2}|\varphi_{k-1}|^2 \\
& + s^n \iint_Q e^{-2s\alpha}\gamma(t)^n\xi_0|\nabla\varphi_{k-1}|^2 \Bigg) \\
& + C\sum_{j=1}^{k-1} s^{3(m-j)+2} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-j)+2}\xi_0|\varphi_j|^2.
\end{aligned}
$$

Now, reasoning as we did with $K_1^{(3)}$, it is not difficult to deduce

$$
\begin{aligned}
K_1^{(4)} &= -\sum_{j=1}^{k-1} \iint_Q u\xi_0\varphi_{k-1}a_{jk}\varphi_j - \iint_Q u\xi_0\varphi_{k-1}a_{kk}\varphi_k \\
&\leq Cs^{n+2} \iint_Q e^{-2s\alpha}\gamma(t)^{n+2}\xi_0|\varphi_{k-1}|^2 \\
&+ \frac{1}{2}\sum_{j=1}^{k-1} s^{3(m-k)-1} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-k)-1}|a_{jk}|^2\xi_0|\varphi_j|^2 \\
&+ \delta\tilde{L}(l,\varphi_k) + \frac{C}{\delta}s^l \iint_Q e^{-2s\alpha}\gamma(t)^l|a_{kk}|^2\xi_0|\varphi_{k-1}|^2,
\end{aligned}
$$

with $\delta > 0$. Again, $\|a_{jk}\|_\infty^2 \leq C(s\gamma(t))^{3(k-j)+3}$ for every $s \geq CT^2\|a_{ji}\|_\infty^{\frac{2}{3(k-j)+3}}$.

This implies that

$$
\begin{aligned}
|K_1^{(4)}| \leq{}& Cs^{n+2} \iint_Q e^{-2s\alpha}\gamma(t)^{n+2}\xi_0|\varphi_{k-1}|^2 \\
&+ C\sum_{j=1}^{k-1} s^{3(m-j)+2} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-j)+2}\xi_0|\varphi_j|^2 + \delta\tilde{L}(l,\varphi_k) \\
&+ \frac{C}{\delta}s^{l+3} \iint_Q e^{-2s\alpha}\gamma(t)^{l+3}\xi_0|\varphi_{k-1}|^2.
\end{aligned}
$$

As said above, if $k = m$ then $K_1^{(5)} \equiv 0$. If $k \leq m-1$, we have,

$$
\begin{aligned}
K_1^{(5)} &= -\iint_Q u\xi_0\varphi_{k-1}a_{k+1,k}\varphi_{k+1} \leq \frac{\varepsilon}{12}s^{3(m-k)} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-k)}|\varphi_{k+1}|^2 \\
&+ \frac{3M^2}{\varepsilon}s^{n+1} \iint_Q e^{-2s\alpha}\gamma(t)^{n+1}\xi_0|\varphi_{k-1}|^2.
\end{aligned}
$$

Altogether we get

$$
\left\{
\begin{aligned}
|K_1| \leq{}& 3\delta\tilde{L}(l,\varphi_k) + \frac{\varepsilon}{12}\left[\mathcal{I}(3(m+1-k),\varphi_k) + \mathcal{I}(3(m-k),\varphi_{k+1})\right] \\
&+ C(\delta,\varepsilon)s^J \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^J|\varphi_{k-1}|^2 \\
&+ C(\varepsilon)s^R \iint_Q e^{-2s\alpha}\gamma(t)^R\xi_0|\nabla\varphi_{k-1}|^2 \\
&+ C\sum_{j=1}^{k-2} s^{3(m-j)+2} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-j)+2}\xi_0|\varphi_j|^2
\end{aligned}
\right. \tag{22}
$$

with $J = J(k) = \max\{l+4, n+2, 3(m-k)+5\}$, $R = \max\{n, l+1\}$ and $n = 2l - 1 - 3(m-k)$. Let us remark that in inequality (22) the positive constants $C(\delta,\varepsilon)$ and $C(\varepsilon)$ are given by $C(\delta,\varepsilon) = C\left(1 + \frac{1}{\delta} + \frac{1}{\varepsilon}\right)$ and $C(\varepsilon) = C\left(1 + \frac{1}{\varepsilon}\right)$.

Going back to the term K_2 and integrating by parts, we get

$$
K_2 = -\sum_{i,j=1}^{N} \iint_Q \left(\varphi_k\, \alpha_{ij}^{k-1}\partial_i(u\xi_0)\,\partial_j\varphi_{k-1} + u\,\xi_0\,\alpha_{ij}^{k-1}\partial_i\varphi_k\,\partial_j\varphi_{k-1}\right)
$$

On the other hand, if $s \geq C(T+T^2)$, (see (18) and (19)),

$$
|\partial_i(u\xi_0)| \leq u|\partial_i\xi_0| + Cs^{l+1}e^{-2s\alpha}\gamma(t)^{l+1}|\xi_0 \leq Cs^{l+1}e^{-2s\alpha}\gamma(t)^{l+1}\xi_0^{1/2}.
$$

Therefore

$$
\left\{
\begin{aligned}
|K_2| \leq{}& \delta\tilde{L}(l,\varphi_k) + \frac{C}{\delta}s^{l+2} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{l+2}|\varphi_{k-1}|^2 \\
&+ \frac{\varepsilon}{12}s^{3(m-k)+1} \iint_Q e^{-2s\alpha}\gamma(t)^{3(m-k)+1}|\nabla\varphi_k|^2 \\
&+ \frac{C}{\varepsilon}s^n \iint_Q e^{-2s\alpha}\gamma(t)^n\xi_0|\nabla\varphi_{k-1}|^2.
\end{aligned}
\right. \tag{23}
$$

For the term K_3 we obtain,

$$|K_3| \leq \delta \tilde{L}(l, \varphi_k) + \frac{C}{\delta} \sum_{j=1}^{k-1} s^l \iint_Q e^{-2s\alpha} \gamma(t)^l |a_{j,k-1}|^2 \xi_0 |\varphi_j|^2,$$

with $\delta > 0$. Again, if $s \geq CT^2 ||a_{j,k-1}||_\infty^{\frac{2}{3(k-j)}}$, we can deduce

$$|K_3| \leq \delta \tilde{L}(l, \varphi_k) + \frac{C}{\delta} \sum_{j=1}^{k-1} s^{l+3(k-j)} \iint_Q e^{-2s\alpha} \gamma(t)^{l+3(k-j)} \xi_0 |\varphi_j|^2. \tag{24}$$

We now have that

$$\left\{ \begin{array}{rl} K_4 = & \displaystyle\sum_{j=1}^{k-1} \int_0^T \langle \nabla \cdot (B_{j,k-1} \varphi_j), u \xi_0 \varphi_k \rangle \, dt \\ = & \displaystyle -\sum_{j=1}^{k-1} \left(\iint_Q \nabla(u\xi_0) \cdot B_{j,k-1} \varphi_k \varphi_j + \iint_Q u \xi_0 \nabla \varphi_k \cdot B_{j,k-1} \varphi_j \right) \end{array} \right.$$

Proceeding as before and having in mind that $s \geq CT^2 ||B_{j,k-1}||_\infty^{\frac{2}{3(k-j)-2}}$, we obtain that

$$\left\{ \begin{array}{l} \displaystyle |K_4| \leq \delta \tilde{L}(l, \varphi_k) + \frac{C}{\delta} \sum_{j=1}^{k-1} s^{l+3(k-j)} \iint_{\mathcal{O}_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^{l+3(k-j)} |\varphi_j|^2 \\ \displaystyle \quad + \frac{\varepsilon}{12} s^{3(m-k)+1} \iint_Q e^{-2s\alpha} \gamma(t)^{3(m-k)+1} |\nabla \varphi_k|^2 \\ \displaystyle \quad + \frac{C}{\varepsilon} \sum_{j=1}^{k-1} s^{2l-3(m-2k+j+1)} \iint_Q e^{-2s\alpha} \gamma(t)^{2l-3(m-2k+j+1)} \xi_0 |\varphi_j|^2, \end{array} \right. \tag{25}$$

with $\delta > 0$ to be chosen.

Coming back to (20), putting together inequalities (22), (23), (24) and (25), and choosing the $\delta = 1/12$, we obtain

$$\left\{ \begin{array}{l} \displaystyle \frac{1}{2} \tilde{L}(l, \varphi_k) \leq \frac{\varepsilon}{4} \left[\mathcal{I}(3(m+1-k), \varphi_k) + \mathcal{I}(3(m-k), \varphi_{k+1}) \right] \\ \displaystyle + C(\varepsilon) \left(s^J \iint_{\mathcal{O}_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^J |\varphi_{k-1}|^2 + s^R \iint_Q e^{-2s\alpha} \gamma(t)^R \xi_0 |\nabla \varphi_{k-1}|^2 \right. \\ \displaystyle \qquad \left. + \sum_{j=1}^{k-2} s^{R_{jk}} \iint_{\mathcal{O}_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^{R_{jk}} |\varphi_j|^2 \right), \end{array} \right. \tag{26}$$

with $k \in \{2, ..., m\}$, $C(\varepsilon) = C(1 + 1/\varepsilon)$ and

$$\left\{ \begin{array}{l} J = \max\{l+4, 2l-3(m-k)+1, 3(m-k)+5\}, \\ R = \max\{l+1, 2l-3(m-k)-1\}, \\ R_{jk} = \max\{3(m-j)+2, l+3(k-j), 2l-3(m-2k+j+1)\}. \end{array} \right.$$

Now we are interested in eliminating the term $s^R \displaystyle\iint_Q e^{-2s\alpha}\gamma(t)^R \xi_0 |\nabla\varphi_{k-1}|^2$ in inequality (26). So as to do that, we define $\tilde{u} = s^R e^{-2s\alpha}\gamma(t)^R$ and we will use the equation satisfied by φ_{k-1}:

$$-L_{k-1}\varphi_{k-1} = -a_{k,k-1}\varphi_k + \partial_t\varphi_{k-1} - \sum_{j=1}^{k-1} a_{j,k-1}\varphi_j + \sum_{j=1}^{k-1} \nabla\cdot(B_{j,k-1}\varphi_j)$$

Multiplying this equation by $\tilde{u}\xi_0\varphi_{k-1}$ and integrating by parts in Q we get:

$$\left\{\begin{array}{l} \displaystyle\sum_{i,j=1}^N \iint_Q \tilde{u}\xi_0\alpha_{ij}^{k-1}\partial_i\varphi_{k-1}\partial_i\varphi_{k-1} = -\sum_{i,j=1}^N \iint_Q \alpha_{ij}^{k-1}\partial_i(\tilde{u}\xi_0)\varphi_{k-1}\partial_j\varphi_{k-1} \\ \displaystyle\qquad\qquad\qquad - \iint_Q \tilde{u}\xi_0\varphi_{k-1}a_{k,k-1}\varphi_k \\ \displaystyle + \int_0^T \langle \partial_t\varphi_{k-1} - \sum_{j=1}^{k-1} a_{j,k-1}\varphi_j + \sum_{j=1}^{k-1}\nabla\cdot(B_{j,k-1}\varphi_j), \tilde{u}\xi_0\varphi_{k-1}\rangle\,dt = \sum_{n=1}^5 H_n\,. \end{array}\right.$$

Next, we are going to estimate each term H_i. Using (18) and (19) it is easy to deduce that, if $s \geq C(T+T^2)$, $|\partial_i(\tilde{u}\xi_0)| \leq Cs^{R+1}e^{-2s\alpha}\gamma(t)^{R+1}\xi_0^{1/2}$ and

$$\left\{\begin{array}{l} \displaystyle |H_1| \leq Cs^{R+1}\iint_Q e^{-2s\alpha}\gamma(t)^{R+1}\xi_0^{1/2}|\varphi_{k-1}||\nabla\varphi_{k-1}| \\ \displaystyle\qquad \leq \frac{\widetilde{a}_0}{4}\iint_Q \tilde{u}\xi_0|\nabla\varphi_{k-1}|^2 + Cs^{R+2}\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{R+2}|\varphi_{k-1}|^2. \end{array}\right. \tag{27}$$

If we take $\tilde{\delta} > 0$, we also have:

$$|H_2| \leq \tilde{\delta}\tilde{L}(l,\varphi_k) + \frac{C}{\tilde{\delta}}s^{2R-l}\iint_Q e^{-2s\alpha}\gamma(t)^{2R-l}\xi_0|\varphi_{k-1}|^2. \tag{28}$$

Proceeding as before and using (18), we get

$$|H_3| = \frac{1}{2}\left|\iint_Q \tilde{u}_t\xi_0|\varphi_{k-1}|^2\right| \leq Cs^{R+2}\iint_Q e^{-2s\alpha}\gamma(t)^{R+2}\xi_0|\varphi_{k-1}|^2. \tag{29}$$

In order to estimate H_4 we recall that if $s \geq CT^2||a_{j,k-1}||_\infty^{\frac{2}{3(k-1-j)+3}}$ then $\|a_{j,k-1}\|_\infty^2 \leq C\left(s\gamma(t)\right)^{3(k-j)}$. Thus, it is no difficult to check

$$\left\{\begin{array}{l} \displaystyle\iint_Q \left|\tilde{u}\xi_0 a_{k-1,k-1}\varphi_{k-1}^2\right| \leq C\iint_Q e^{-2s\alpha}\left[s\gamma(t)\right]^{R+3/2}\xi_0|\varphi_{k-1}|^2, \\ \displaystyle\iint_Q \left|\tilde{u}\xi_0 a_{j,k-1}\varphi_j\varphi_{k-1}\right| \leq C\iint_Q e^{-2s\alpha}\left[s\gamma(t)\right]^{R+3/2}\xi_0|\varphi_{k-1}|^2 \\ \displaystyle\qquad\qquad\qquad\qquad + C\iint_Q e^{-2s\alpha}\left[s\gamma(t)\right]^{R-3/2+3(k-j)}\xi_0|\varphi_j|^2. \end{array}\right.$$

Therefore

$$|H_4| \le C s^{R-3/2+3(k-j)} \sum_{j=1}^{k-1} \iint_Q e^{-2s\alpha}\gamma(t)^{R-3/2+3(k-j)}\xi_0|\varphi_j|^2. \tag{30}$$

Integrating by parts in H_5 and taking into account assumption (18) and (19), and that $\|B_{j,k-1}\|_\infty^2 \le C\left(\frac{s}{T^2}\right)^{3(k-j)-2}$, we can reason as before and obtain

$$\begin{cases} \quad |H_5| \le C\displaystyle\sum_{j=1}^{k-1} s^{R-3/2+3(k-j)} \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{R-3/2+3(k-j)}|\varphi_j|^2 \\ +\displaystyle\sum_{j=1}^{k-1} s^{R+3(k-j)-2} \iint_Q e^{-2s\alpha}\gamma(t)^{R+3(k-j)-2}\xi_0|\varphi_j|^2 + \frac{\widetilde{a}_0}{4}\iint_Q \tilde{u}\xi_0|\nabla\varphi_{k-1}|^2. \end{cases} \tag{31}$$

Summarizing, taking into account (3), adding (27)–(31) and using again (18), we deduce:

$$\begin{cases} \dfrac{\widetilde{a}_0}{2}\displaystyle\iint_Q \tilde{u}\xi_0|\nabla\varphi_{k-1}|^2 \le C s^{R+2}\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{R+2}|\varphi_{k-1}|^2 \\ \qquad +\tilde{\delta}\tilde{L}(l,\varphi_k) + \dfrac{C}{\tilde{\delta}}s^{2R-l}\displaystyle\iint_Q e^{-2s\alpha}\gamma(t)^{2R-l}\xi_0|\varphi_{k-1}|^2 \\ \quad +C\displaystyle\sum_{j=1}^{k-1} s^{R-3/2+3(k-j)}\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\gamma(t)^{R-3/2+3(k-j)}|\varphi_j|^2. \end{cases} \tag{32}$$

Finally, if we now choose $\tilde{\delta} = \widetilde{a}_0/4C(\varepsilon)$, with $C(\varepsilon)$ the constant appearing in (26), i.e., $\tilde{\delta} = \frac{\widetilde{a}_0\varepsilon}{4C(\varepsilon+1)}$, and we combine (32) with (26), we obtain the proof of the result. □

3. Null controllability of system (4). Proof of Theorem 1.2

We will devote this section to show Theorem 1.2. As said above, it is well known that Theorem 1.2 is equivalent to the observability inequality (9) satisfied by the solutions φ of the adjoint system (8) and, in fact, the constant $e^{C\,\mathcal{H}}$ appearing in (11) coincides with C_0, the constant appearing in (9), (see for instance [9]). We establish this observability inequality in the next result:

Proposition 3.1. *Under the assumptions of Theorem* 1.1*, there exists a constant* $C > 0$ *(which only depends on* Ω*,* ω_0*,* $\widetilde{a}_0$*,* $\widetilde{M}_0$*,* a_0 *and* M_0*) such that, for every* $\varphi_0 \in L^2(\Omega)^m$*, the corresponding solution* φ *to* (8) *satisfies*

$$||\varphi(\cdot,0)||^2_{L^2(\Omega)} \le e^{C\,\mathcal{H}}\iint_{\omega_0\times(0,T)} |\varphi_1(x,t)|^2\,dx\,dt, \tag{33}$$

where $\mathcal{H}$ *is given in Theorem* 1.2.

Proof: The proof of this result is standard (see e.g., [9], [6] and [12]) and it is a consequence of (10) and the energy inequality satisfied by the solutions to (8):

$$||\varphi(\cdot,t_1)||^2_{L^2(\Omega)^m} \leq e^{C[1+\max_{i\leq j}(||a_{ij}||_\infty+||B_{ij}||^2_\infty)](t_2-t_1)}||\varphi(\cdot,t_2)||^2_{L^2(\Omega)^m}\,,$$

where $0 \leq t_1 \leq t_2 \leq T$ and $C > 0$ is a new constant which depends on m, $\widetilde{a}_0$ and $M_0 = \max_{2\leq i\leq m}||a_{i,i-1}||_\infty$.

If φ is a solution to (8), from the energy inequality, we deduce

$$\sum_{i=1}^m ||\varphi_i(\cdot,0)||^2_{L^2(\Omega)} \leq \frac{2}{T}\, e^{C[1+\max_{i\leq j}(||a_{ij}||_\infty+||B_{ij}||^2_\infty)]T} \sum_{i=1}^m \int_{T/4}^{3T/4}\int_\Omega |\varphi_i|^2. \qquad (34)$$

On the other hand, taking into account (10) and (18), one has

$$s^3\sum_{i=1}^m \int_{T/4}^{3T/4}\int_\Omega e^{-2s\alpha}\gamma(t)^3|\varphi_i|^2 \leq Cs^l \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\gamma(t)^l|\varphi_1|^2,$$

for a new constant $C = C(\Omega,\omega_0,a_0,M) > 0$ and for every $s \geq s_0$. Secondly, it is not difficult to see

$$\begin{cases} s^3e^{-2s\alpha}\gamma(t)^3 \geq \dfrac{2^{12}}{3^3}s^3T^{-6}\exp\left(\dfrac{-2^5M_0s}{3T^2}\right) \\ \qquad\qquad\qquad \geq \dfrac{1}{3^3}\left(\dfrac{2l}{m_0}\right)^3\exp\left(\dfrac{-2^5M_0s}{3T^2}\right),\ \forall (x,t)\in\Omega\times(T/4,3T/4), \\ s^le^{-2s\alpha}\gamma(t)^l \leq s^l2^{2l}T^{-2l}\exp\left(\dfrac{-2^3m_0s}{T^2}\right) \leq \left(\dfrac{l}{2em_0}\right)^l, \quad \forall (x,t)\in Q, \end{cases}$$

if we choose $s \geq (l/8m_0)T^2$, with $m_0 = \min_{\overline{\Omega}}\alpha_0(x) > 0$ and $M_0 = \max_{\overline{\Omega}}\alpha_0(x)$. Now, combining these three last inequalities, we readily deduce

$$\sum_{i=1}^m \int_{T/4}^{3T/4}\int_\Omega |\varphi_i|^2 \leq Ce^{Cs/T^2}\iint_{\omega_0\times(0,T)}|\varphi_1|^2$$

for every $s \geq s_1 = \sigma_1\left[T + T^2 + T^2\max_{i\leq j}\left(||a_{ij}||_\infty^{\frac{2}{3(j-i)+3}} + ||B_{ij}||_\infty^{\frac{2}{3(j-i)+1}}\right)\right]$ and $\sigma_1 = \max\{\sigma_0,(l/8m_0)\}$. Finally, by setting $s = s_1$ in the previous estimate and by recalling (34), we end the proof. □

4. Further results and comments

We will finalize this work doing some remarks and establishing some additional results.

1. It is possible to prove the null controllability of system (4) at time T if we replace (7) with the hypothesis

$$a_{i,i-1} \geq a_0 > 0 \text{ or } -a_{i,i-1} \geq a_0 > 0 \text{ in } \omega_0 \times (T_0, T_1), \quad \forall i : 2 \leq i \leq m \qquad (35)$$

with $0 \leq T_0 < T_1 \leq T$. Indeed, let $\widehat{y} \in C^0([0,T]; L^2(\Omega)^m)$ be the solution to (4) for $v \equiv 0$. Hypothesis (35) allows us to prove the existence of a control $\widehat{v} \in L^2(\Omega \times (T_0, T_1))$ which drives system (4) (posed in the cylinder $\Omega \times (T_0, T_1)$) from the initial data $\widehat{y}(\cdot, T_0)$ (at time T_0) to the rest at time T_1. Now, if we take $v \equiv 0$ in the set $(0, T_0) \cup (T_1, T)$ and $v \equiv \widehat{v}$ on the interval (T_0, T_1), we have that the solution y to (4) corresponding to the control $v \in L^2(Q)$ satisfies (5). Moreover, the control v can be chosen such that (11) holds with $\mathcal{H}$ given by

$$\mathcal{H} \equiv 1 + T_1 + \frac{1}{T_1 - T_0} + \max_{i \leq j} \left(||a_{ij}||_\infty^{\frac{2}{3(j-i)+3}} + ||B_{ij}||_\infty^{\frac{2}{3(j-i)+1}} + T_1(||a_{ij}||_\infty + ||B_{ij}||_\infty^2) \right).$$

On the other hand, let us assume that A and B are complete matrices which satisfy (35) and

$$\begin{cases} a_{ij} \equiv 0 \text{ in } \omega_0 \times (T_0, T_1), & \forall i, j : i \geq j + 2, \\ B_{ij} \equiv 0 \text{ in } \omega_0 \times (T_0, T_1), & \forall i, j : i \geq j + 1, \end{cases}$$

for a nonempty open subset $\omega_0 \subset \omega$ and $T_0, T_1 \in [0, T]$ with $T_0 < T_1$. Then, it is also not difficult to check that Theorem 1.2 is still valid and Theorem 1.1 holds with

$$\sum_{i=1}^{m} \widetilde{\mathcal{I}}(3(m+1-i), \varphi_i) \leq C_0 s^l \iint_{\omega_0 \times (T_0, T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^l |\varphi_1|^2,$$

instead of (10). In the previous inequality $\widetilde{\alpha}(x,t)$, $\widetilde{\gamma}(t)$ and $\widetilde{\mathcal{I}}(d,z)$ are given by: $\widetilde{\alpha}(x,t) = \alpha_0(x)/(t - T_0)(T_1 - t)$, $\widetilde{\gamma}(t) = ((t - T_0)(T_1 - t))^{-1}$ and

$$\widetilde{\mathcal{I}}(d, z) \equiv s^{d-2} \iint_{\Omega \times (T_0, T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{d-2} |\nabla z|^2 + s^d \iint_{\Omega \times (T_0, T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^d |z|^2.$$

2. As a direct consequence of the Carleman inequality (10) we obtain the unique continuation property:

> *"Under assumptions of Theorem* 1.1*, if* $\varphi \in C^0([0,T]; L^2(\Omega)^m)$ *is solution to (*8*) and satisfies* $\varphi_1(x,t) = 0$ *in* $\omega_0 \times (0,T)$*, then,* $\varphi \equiv 0$ *in* Q*".*

It is well known that this unique continuation property for the adjoint problem (8) implies the approximate controllability property at time T of system (4).

When $m = 2$ and $L_1 = L_2$, i.e., for two equations, it is proved in [16], that the unique continuation property for the adjoint problem (8) is valid even when

$a_{21} = 0$ in ω but $a_{21} \neq 0$ in $\widetilde{\omega}$ an open subset of Ω. The problem remains open for $L_1 \neq L_2$.

3. Theorems 1.1 and 1.2 can readily be generalized to the case where $v = (v_1, v_2, \ldots, v_r)^*$ (r control forces, $r \geq 1$) and the coupling and control matrices A, B and D satisfies: $B \in L^\infty(Q)^{Nm^2}$ is as in (6), $A \in L^\infty(Q)^{m^2}$ is given by

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{pmatrix}, \; A_{ii} = \begin{pmatrix} a^i_{11} & a^i_{12} & a^i_{13} & \cdots & a^i_{1,s_i} \\ a^i_{21} & a_{22} & a^i_{23} & \cdots & a^i_{2,s_i} \\ 0 & a^i_{32} & a^i_{33} & \cdots & a^i_{3,s_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a^i_{s_i,s_i-1} & a^i_{s_i,s_i} \end{pmatrix}$$

with $s_i \in \mathbb{N}$, $\sum_{i=1}^r s_i = m$, $a^i_{j,j-1}$ satisfying (7) for every (i,j) ($1 \leq i \leq r$, $2 \leq j \leq s_i$), and $D \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^m)$ such that $D = (\, e_{S_1} \,|\, e_{S_2} \,|\, \cdots \,|\, e_{S_r})$ with $S_i = 1 + \sum_{j=1}^{i-1} s_j$, $1 \leq i \leq r$ (e_j is the j-th element of the canonical basis of $\mathbb{R}^m$). Observe that matrix A do not satisfy (7) for $i = S_2, ..., S_r$ and even so, under the previous assumptions, a null controllability result for system (4) can be proved if we add a control in each equation where (7) does not hold. Indeed, let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)^*$ be a solution to the adjoint problem (8). Thanks to the structure of the coupling matrices A and B, we can apply Lemma 2.2 to φ_k, for every $k \neq S_i$ with $1 \leq i \leq r$, and obtain, from (15),

$$\sum_{i=1}^m \mathcal{I}(3(m+1-i), \varphi_i) \leq C_0 \sum_{i=1}^r s^{l_i} \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^{l_i} |\varphi_{S_i}|^2, \quad \forall s \geq s_0,$$

with s_0 as in Theorem 1.1, $C_0 = C_0(\Omega, \omega_0, \widetilde{a}_0, \widetilde{M}_0, a_0, M_0) > 0$ and $l_i \geq 3$ ($1 \leq i \leq r$). Theorem 1.2 is a consequence of this last inequality.

4. Following the ideas of [4] (see Theorem 1.3.3, p. 156), from the Carleman inequality (10) we can prove the null controllability of system (4) with controls in $L^\infty(Q)$. To be precise, it is possible to show

Theorem 4.1. *Under hypotheses of Theorem* 1.1, *there exists a control* $v \in L^\infty(Q)$ *satisfying*

$$||v||^2_\infty \leq e^{C\,\mathcal{H}} \sum_{i=1}^m ||y_{0,i}||^2_{L^2(\Omega)},$$

such that $\operatorname{Supp} v \subset \overline{\omega}_0 \times [0,T]$ *and the corresponding solution* y *to* (4) *satisfies* (5). *In this inequality,* C *is a positive constant only depending on* Ω, ω_0, $\widetilde{a}_0$, $\widetilde{M}_0$, a_0 *and* M_0, *and* $\mathcal{H}$ *is as in Theorem* 1.2. □

Sketch of the proof: From (10) and using the regularizing effect of problem (8), it is possible to prove the refined Carleman inequality for the solutions $\varphi = (\varphi_1, ..., \varphi_m)^*$ to (8) (valid for $s \geq s_0$, with s_0 given in Theorem 1.1):

$$||(s\gamma(t))^{-K} e^{-s\alpha} \varphi||^2_\infty + \sum_{i=1}^m \mathcal{I}(3(m+1-i), \varphi_i) \leq C_1 s^l \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \gamma(t)^l |\varphi_1|^2,$$

where $C_1 = C_1(\Omega, \omega_0, \widetilde{a}_0, \widetilde{M}_0, a_0, M_0) > 0$ and $K = K(N, m) \in \mathbb{N}$ are two new constants. A duality argument allows us to obtain the proof from this refined Carleman inequality. □

Theorem 4.1 is crucial in order to study the exact controllability to the trajectories of cascade nonlinear parabolic systems when the nonlinearities considered depend on the state and its gradient and have a superlinear growth at infinity (for the proof, see [13]).

5. When the coefficients a_{ij} and B_{ij} of system (4) are regular enough (for instance, if they are constants or only depend on t), it is possible to show Theorem 1.2 using the strategy of fictitious controls developed in [11] and [12]. Briefly, this technique consists in introducing a control function in each equation of our system (and, therefore, the null controllability of the system is a consequence of the Carleman inequality (15)) and, subsequently, eliminate the $m-1$ fictitious controls using the cascade structure of the system (see [12]). This strategy cannot be applied in the case of system (4) due to a lack of regularity of the coefficients.

6. *Boundary controls:* In view of known controllability results for a linear heat equation, it would be natural to wonder whether the null controllability result for system (4) remain valid when one considers one control force exerted on the boundary: $y = e_1 v 1_\gamma$ on Σ, where $\gamma \subset \partial\Omega$ is a relative open subset of $\partial\Omega$. Nevertheless, there exist negative results for some 1-d cascade linear coupled parabolic systems with $m = 2$ (cf. [7]), which reveals the different nature of the controllability properties for a single heat equation and for coupled parabolic systems.

7. In the present work we have provided a sufficient condition on the matrices A, B and D which ensures the null controllability of system (4) at time T. Let us observe that when $B \equiv 0$ and A is a constant matrix, under assumptions (6) and (7), the exact controllability of the ordinary differential system

$$\begin{cases} y' + Ay = Dv \text{ in } [0,T], \\ y(0) = y_0 \in \mathbb{R}^N. \end{cases}$$

holds with $D \equiv e_1$ since one has the so-called Kalman rank condition

$$\operatorname{rank}\,[\, D \,|\, AD \,|\, A^2 D \,|\, \dots \,|\, A^{m-1} D \,] = m.$$

Thus, it would be very interesting to try to generalize this condition to the case of coupled parabolic system like (4) and give a condition on the matrices A, B and D which is equivalent to the null controllability at time T of system (4). At the moment, the general problem is open but some results have been recently obtained in [3]. □

References

[1] F. Ammar Khodja, A. Benabdallah, C. Dupaix, I. Kostin, Controllability to the trajectories of phase-field models by one control force. *SIAM J. Control Optim.,* **42** (2003), no. 5, pp. 1661–1680.

[2] F. Ammar Khodja, A. Benabdallah, C. Dupaix, Null-controllability of some reaction-diffusion systems with one control force. *J. Math. Anal. Appl.* **320** (2006), no. 2, pp. 928–943.

[3] F. Ammar Khodja, A. Benabdallah, C. Dupaix, M. González-Burgos, Controllability for a class of reaction-diffusion systems: generalized Kalman's condition. *C. R. Math. Acad. Sci. Paris* **345** (2007), no. 10, 543–548.

[4] V. Barbu, Controllability of parabolic and Navier-Stokes equations. *Sci. Math. Jpn.* **56** (2002), no. 1, 143–211.

[5] O. Bodart, M. González-Burgos, R. Pérez-García, Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity. *Comm. PDE.* **29** (2004), no. 7–8, 1017–1050.

[6] A. Doubova, E. Fernández-Cara, M. González-Burgos, E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. *SIAM J. Control Optim.* **41** (2002), no. 3, 798–819.

[7] E. Fernández-Cara, M. González-Burgos, L. de Teresa Boundary controllability of parabolic coupled equations. Submitted.

[8] E. Fernández-Cara, S. Guerrero, O.Yu. Imanuvilov, J.P. Puel, On the controllability of the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls. *C. R. Math. Acad. Sci. Paris, Ser. I* **340** (2005), no. 4, 275–280.

[9] E. Fernández-Cara, E. Zuazua, The cost of approximate controllability for heat equations: The linear case. *Adv. Differential Equations* **5** (2000), 465–514.

[10] A. Fursikov, O. Yu. Imanuvilov, *Controllability of Evolution Equations.* Lecture Notes Ser. 34, Seoul National University, Korea, 1996.

[11] M. González-Burgos, R. Pérez-García, Controllability of some coupled parabolic systems by one control force. *C. R. Math. Acad. Sci. Paris, Ser. I* **340** (2005), no. 2, 125–130.

[12] M. González-Burgos, R. Pérez-García, Controllability results for some nonlinear coupled parabolic systems by one control force. *Asymptotic Analysis* **46** (2006) 123-162.

[13] M. González-Burgos, R. Pérez-García, L. de Teresa, Controllability results for some nonlinear cascade parabolic systems. In preparation.

[14] O. Yu. Imanuvilov, Controllability of parabolic equations, *Mat. Sb.* **186** (1995), 102–132.

[15] O. Yu. Imanuvilov, M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations. *Publ. Res. Inst. Math. Sci.* **39** (2003), no. 2, 227–274.

[16] O. Kavian, L. de Teresa, Unique continuation principle for systems of parabolic equations. To appear in ESAIM:COCV.

[17] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur. *Comm. PDE.* **20** (1995), no. 1–2, 335–356.

[18] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Rev.* **20** (1978), no. 4, 639–739.

[19] L. de Teresa, Insensitizing controls for a semilinear heat equation. *Comm. PDE.* **25** (2000), no. 1–2, 39–72.

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